Schur's inequality

If a, b, c are all positive and $n \ge 0$ or $n \le -1$, then

$$a^n (a-b)(a-c) + b^n (b-a)(b-c) + c^n (c-a)(c-b) \ge 0$$

Solution

(1) First let $n > 0$.

Without lost of generality, we may assume $\mathbf{a \ge b \ge c}$.

$\therefore$ $a^n \ge b^n \ge c^n$.

$\therefore$ $a^n (a-b)(a-c) \ge b^n (a-b)(b-c)$ and $c^n (c-a)(c-b) \ge 0$

$\therefore$ $a^n (a-b)(a-c) + c^n (c-a)(c-b) \ge b^n (a-b)(b-c)$

$\therefore$ $a^n (a-b)(a-c) - b^n (a-b)(b-c) + c^n (c-a)(c-b) \ge 0$

$\therefore$ $a^n (a-b)(a-c) + b^n (b-a)(b-c) + c^n (c-a)(c-b) \ge 0$

(2) If $n = 0$

$$\sum_{a,b,c} a^n (a-b)(a-c) = \sum_{a,b,c} (a-b)(a-c) = \sum_{a,b,c} a^2 - \sum_{a,b,c} bc$$

$$= \frac{1}{2}\sum_{a,b,c} (a-b)^2 \ge 0$$

(Note that $\sum_{a,b,c} a^2 = \sum_{a,b,c} b^2 = \sum_{a,b,c} c^2$ and $\sum_{a,b,c} ab = \sum_{a,b,c} bc = \sum_{a,b,c} ca$)

(3) If $n \le -1$,

Let $n = -m - 1$ then $m = -n - 1 \ge 0$

Put $a = \frac{1}{\alpha}$, $b = \frac{1}{\beta}$, $c = \frac{1}{\gamma}$,

then $$\sum_{a,b,c} a^n (a-b)(a-c) = \sum_{\alpha,\beta,\gamma} \left(\frac{1}{\alpha}\right)^n \left(\frac{1}{\alpha} - \frac{1}{\beta}\right)\left(\frac{1}{\alpha} - \frac{1}{\gamma}\right)$$

$$= \sum_{\alpha,\beta,\gamma} \alpha^{-n} \left(\frac{\beta-\alpha}{\alpha\beta}\right)\left(\frac{\gamma-\alpha}{\alpha\gamma}\right)$$

$$= \frac{1}{\alpha\beta\gamma}\sum_{\alpha,\beta,\gamma} \alpha^m (\alpha-\beta)(\alpha-\gamma) \ge 0, \quad \text{by (1) and (2).}$$